Announcements

Upcoming plans:

Today: set Cover and approximating it

Monday: solving NP-hard problems exactly

Wed-Friday: using hardness for crypto (lectures by Noah Stephens-Davidowitz)

hw 8 due today
hw 9 out
prelim regrade requests due today

Bowers CIS Pre-Enroll Event

CornellBowers
College of Computing + Information Science

WICC URMC Queery issa

Open to all CS/IS/ISST pre-majors, majors, minors, and anyone interested in computing.

→ April 13, 2026
4:00 – 5:30 PM
CIS Building, Wayfair Commons (Room 132), Conine and Shah Families Active Learning Classroom (Room 142)

WICC, URMC, ACSU, ISSA, Queery

The Set Cover problem

sets $S_1 \ldots S_n$ $\quad U = \bigcup_i S_i$

weights $w_1 \quad w_n$

goal select $I \subseteq \{1, \ldots, n\}$ $\quad \bigcup_{i \in I} S_i = U$

& $\sum_{i \in I} w_i$ minimal

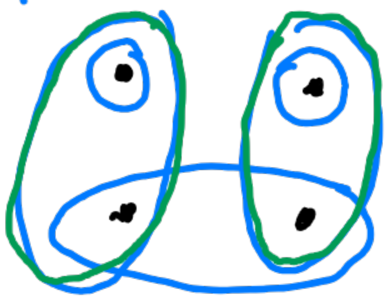
(in NP version $w_i = 1$ all i & need $|I| \leq k$)

Claim Set cover NP-hard

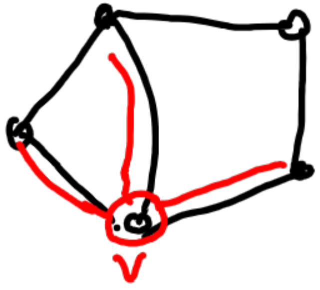
Vertex Cover $\leq$ Set cover

Example

U



$w_i = 1 \; \forall i$



reduction

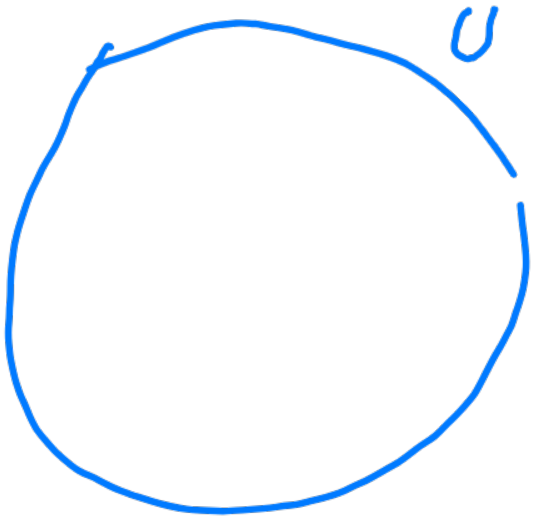
$\forall v \in V \longrightarrow S_v = \{e : \text{adjacent to } v\}$

$w_v = 1 \quad \forall v$

vertex cover yes if & only if min set cover weight $\leq k$

Greedy ideas

to approximate set cover solution



input $S_1 \dots S_k$

$w_1 \dots w_k$

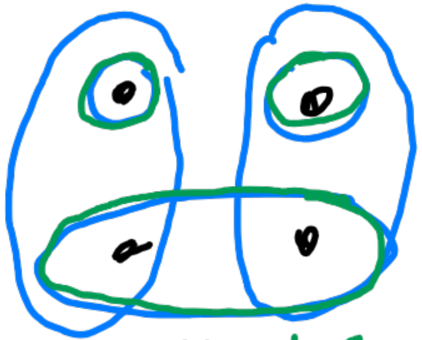
- take smallest weight

$w_0 = 2$

$w_i = 1$

- take big sets $|S_i|$ — $w_0 = n^2 \max_{i \neq 0} w_i$
- take min $w_i / |S_i|$ (weight per element covered)

Example



$w_i = 1 \ \forall i$

step 1: $\frac{w_i}{|S_i|} = \frac{1}{2}$

step 2: $\frac{w_i}{\#\text{ newly covered}} = 1$

step 3

$w_i = 1 - \varepsilon$

Opt = 2, greedy 3

Algorithm

$R = U$ (uncovered elements)

$I = \emptyset$

While $R \neq \emptyset$

select i min $\frac{w_i}{|S_i \cap R|}$ ← # new elements covered

add i to I

$R \leftarrow R \setminus S_i$

endwhile

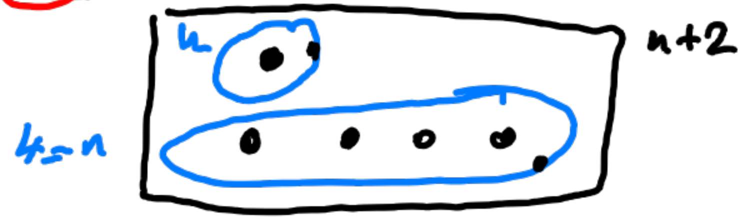
Join by Web **PollEv.com/evatardos772**

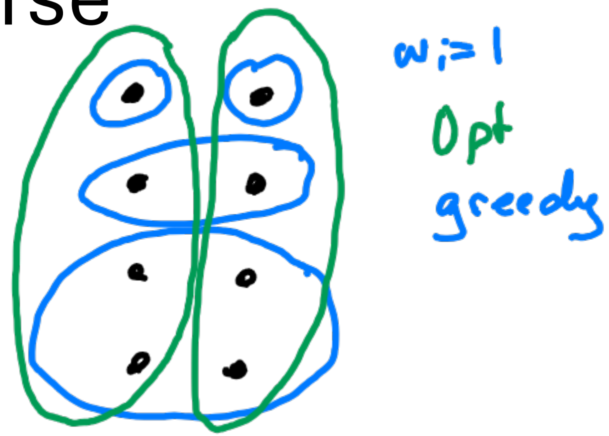
Is ratio based greedy a 1.5 approximation for set cover?

A. Yes, seems that way

B. No: the ratio can be worse

$R = U$, $I = \emptyset$
while $R \neq \emptyset$
 Select i min $\frac{w_i}{|S_i \cap R|}$
 add i to I
 $R = R \setminus S_i$
endwhile





Measuring quality as we proceed…

element $u \in U$ covered first set S_i

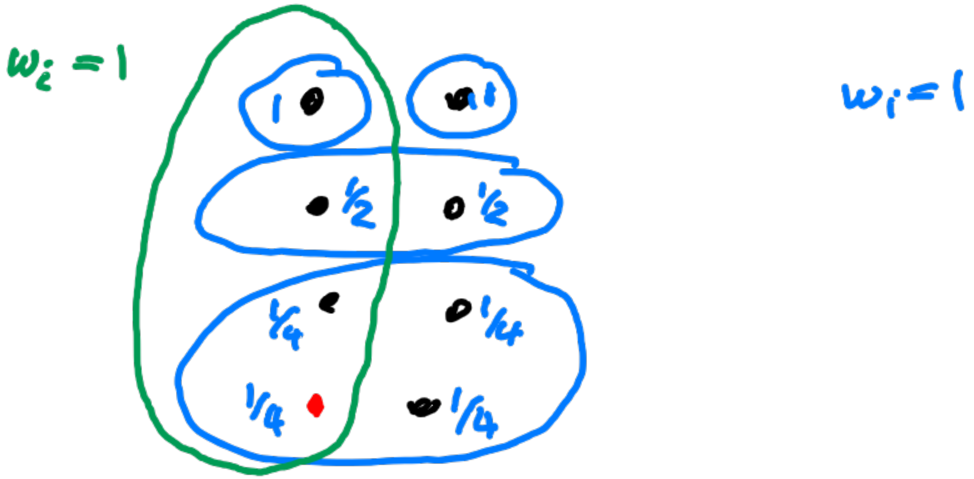
$$c_u = \frac{w_i}{|S_i \cap R|} \quad \text{when covered}$$

$\boxed{\text{Fact 1:}}$ greedy set I

$$\sum_{i \in I} w_i = \sum_{u \in U} c_u$$

Proof: · when adding set i we divide up w_i among new covered elements

$R = U$, $I = \emptyset$
while $R \neq \emptyset$
 Select i min $\frac{w_i}{|S_i \cap R|}$
 add i to I
 $R = R \setminus S_i$
endwhile

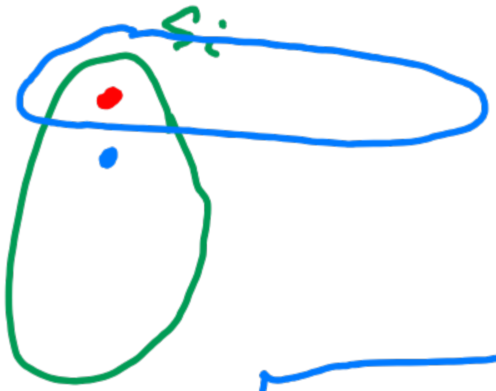


Comparing the total cost-shared on sets

Any set S_i ? w_i compare to $\sum_{u \in S_i} c_u$

possible maybe

what greedy paid



- first element covered in the set $\leq \frac{w_i}{|S_i|}$

now $R \cap S_i = \emptyset$

$\Rightarrow$ choice greedy better than S_i

- second element covered in the set $\leq \frac{w_i}{|S_i|-1}$

now $|R \cap S_i| \leq 1$

- k+1 element $\leq \frac{w_i}{|S_i|-k}$

...

Summary $\boxed{\text{Fact 2}}$

$$\sum_{u \in S_i} c_u \leq \frac{w_i}{|S_i|} + \frac{w_i}{|S_i|-1} + \dots + \frac{w_i}{1}$$

Proving the Approximation Guarantee

Comparing I^* optimal solution to I greedy

$$\sum_{i\in I} w_i \underset{\text{Fact 1}}{=} \sum_{u\in U} c_u \underset{I^* \text{ is set cover}}{\le} \sum_{i\in I^*} \sum_{u\in S_i} c_u \underset{\text{Fact 2}}{\le} \sum_{i\in I^*} w_i \left(\frac{1}{|S_i|} + \frac{1}{|S_i|-1} + \dots + 1\right)$$

$$\le \sum_{i\in I^*} w_i \left(1 + \frac{1}{2} + \dots + \frac{1}{d}\right)$$

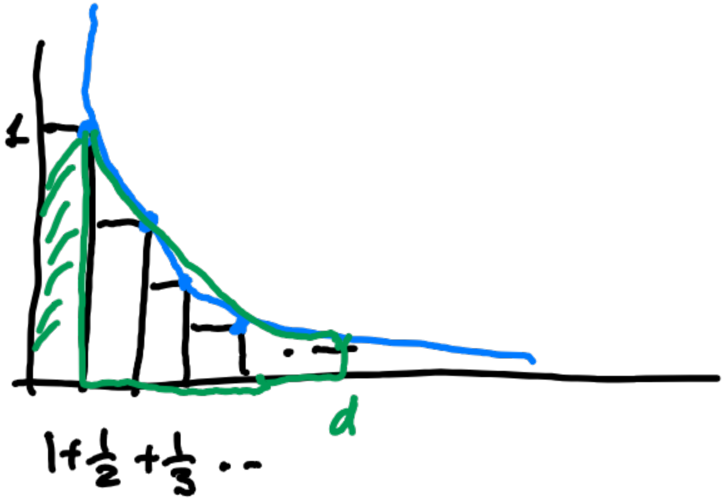
I^*

if $d = \max_i |S_i|$

Note $1 + \frac{1}{2} + \dots + \frac{1}{d} = H_d$ harmonic number

Notes on the quality of this approximation

How big is $H_d = 1 + \frac{1}{2} + \cdots \frac{1}{d}$



$$\sim \int_1^d \frac{1}{x}\, dx \sim \ln d$$

Fact: $\ln(d+1) \leq H_d \leq 1 + \ln d$

Special case vertex cover

max $|S_v|$ = degree in graph

Unless $P = NP$ $O(\ln d)$ best approximation possible